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ORIGINAL PAPER

A variational approach for a class of nonlocal elliptic boundary value problems

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Abstract The aim of this paper is to apply the variational iteration method to a class of nonlinear, nonlocal, elliptic boundary value problems. The uniform convergence of the scheme is presented and the work is illustrated by considering a number of test examples that confirm the accuracy and efficacy of the iterative process. The computational results show that the scheme is reliable, converges fast and compares very well with the existing analytic solutions.

Keywords Variational iteration method · Nonlinear, non-local, elliptic boundary value problems · Uniform convergence

1 Introduction

The goal of this article is to manipulate the variational iteration method (VIM) for obtaining accurate numerical solutions for a class of one-dimensional nonlocal elliptic boundary value problems. The first is the non-homogenous problem:

$$-\alpha \left(\int_0^1 u(t)\, dt \right) u''(x) = f(x), \tag{1.1}$$

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with boundary conditions

$$u(0) = a, \quad u(1) = b. \tag{1.2}$$

The second problem is nonlinear and homogeneous which is given by

$$-\alpha \left(\int_0^1 u(t)\, dt \right) u''(x) + [u(x)]^{2m+1} = 0, \tag{1.3}$$

where m is a positive integer, $x \in (0, 1)$, and complimented with the same boundary conditions as those given in (1.2).

These nonlocal elliptic problems and other similar ones [1,2] arise in several physical models including system of particles in thermodynamical equilibrium interacting via gravitational (Coulomb) potential, Ohmic heating with variable thermal conductivity, fully turbulent behavior of real flow, shear bands in metals deformed under high strain rates, local model for the temperature of a thin region which occurs during linear friction welding, thermo-viscoelastic flows, and one-dimensional fluid flows with rate of strain proportional to a power of stress multiplied by a function of temperature. For details on such applications see [10] and the references therein. Several papers dealt with such elliptic problems, for instance in [1], Cannon and Galiffa developed a numerical method for the Eq. (1.3) with boundary conditions (1.2), in which they established a priori estimates and the existence and uniqueness of the solution to the nonlinear auxiliary problem via the Schauder fixed point theorem. They proved the existence and uniqueness to the problem and analyzed a discretization of the above problem and showed that a solution to the nonlinear difference problem exists and is unique and that the numerical procedure converges with error $O(h)$.

The strategy in this paper is to present the VIM to acquire accurate numerical solutions for the nonlocal elliptic problems given in (1.1) and (1.3) which are complimented with the boundary conditions (1.2). In recent years, numerous papers focused on implementing the (VIM) as a powerful method for the exact and/or numerical solution of a wide spectrum of nonlinear equations [3] including algebraic, differential, partial-differential, functional-delay and integro-differential equations (see [4–9,11–14] and the references therein). To demonstrate convergence and accuracy characteristics of the VIM strategy, a number of test examples are included. The numerical experiments confirm the reliability of the approach as it handles such nonlocal elliptic differential equations without imposing limiting assumptions that could change the physical structure of the solution. The basic strategy of the procedure relies on constructing a correction functional using a general Lagrange multiplier, which is chosen in a suitable manner that its correction solution is improved with respect to the initial approximation or to the trial function. The solution that arises from this iterative method is in the form of successive approximations that yield the exact solution or converge to it.

The outline of the paper is as follows. In Sect. 2, a brief overview of the VIM is given as well as the variational iteration formulation of the nonlocal elliptic problems. In Sect. 3, the uniform convergence of the VIM procedure is presented. In Sect. 4, five test examples are given to verify the accuracy and convergence of the (VIM) strategy. Then in Sect. 5, a conclusion is given that briefly summarizes the results.

2 The numerical method

In this section, we will outline the basic strategy of the VIM when applied to the nonlocal elliptic boundary value problem (1.3)–(1.2). The scheme for Eq. (1.1) follows analogously. Following the approach, we need to construct a correction functional that has the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n(s))_{ss} - (1/\alpha(p))\ \left[\tilde{u}_n(s)\right]^{2m+1}\right] ds, \tag{2.1}$$

where $p = \int_0^1 u(t)\,dt$, while the subscript $n = 0, 1, 2, ...$ denotes the nth order approximation. The optimal value of the Lagrangian multiplier λ will be determined using the variational theory, and $\tilde{u}_n$ is the restricted variation which implies that $\delta(\tilde{u}_n) = 0$.

To start, operating the variation with respect to u_n on both sides of (2.1) yields:

$$\frac{\delta u_{n+1}}{\delta u_n} = 1 + \frac{\delta}{\delta u_n}\left(\int_0^x \lambda(s)\left[(u_n(s))_{ss} - (1/\alpha(p))\ \left[\tilde{u}_n(s)\right]^{2m+1}\right] ds\right), \tag{2.2}$$

The nonlinear restricted term in the integrand vanishes upon taking the variation. As a result, we get

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\left(\int_0^x \lambda(s)\ (u_n)_{ss}\ ds\right). \tag{2.3}$$

Integrating by parts twice, we have

$$\begin{aligned} \delta u_{n+1}(x) = \delta u_n(x) + \delta\left[\lambda(x)(y_n)_s(x)\right] - \delta\left[\lambda'(x) y_n(x)\right] \\ + \delta\left(\int_0^x \lambda''(s)\ u_n(s)\ ds\right), \end{aligned} \tag{2.4}$$

or equivalently

$$\delta u_{n+1}(x) = \left[1 - \lambda'(x)\right]\delta u_n(x) + \delta\left[\lambda(x)(u_n)_s(x)\right] + \delta\left(\int_0^x \lambda''(s)\ u_n(s)\ ds\right). \tag{2.5}$$

Thus, the stationary conditions are:

$$\begin{aligned} &1-\lambda'(s)\big|_{s=x}=0,\\ &\lambda(s)|_{s=x}=0, \\ &\lambda''(s)=0. \end{aligned} \tag{2.6}$$

The solution of Eq. (2.6) results in the following Lagrange multiplier:

$$\lambda(s)=s-x. \tag{2.7}$$

The correction functional for Eq. (1.3) is thus given by

$$u_{n+1}(x)=u_n(x)+\int\limits_0^x (s-x)\left[(u_n(s))_{ss}-(1/\alpha(p_n))\;u_n^{2m+1}(s)\right]ds, \tag{2.8}$$

where

$$p_n=\int\limits_0^1 u_n(t)\;dt. \tag{2.9}$$

Obviously and in a similar manner, the correction functional for (1.1) takes the form

$$u_{n+1}(x)=u_n(x)+\int\limits_0^x (s-x)\left[(u_n(s))_{ss}+(1/\alpha(p_n))\;f(s)\right]ds. \tag{2.10}$$

Consequently, the solution can be obtained from

$$u(x)=\lim_{n\to\infty}u_n(x). \tag{2.11}$$

In other words, the correction functional (2.10) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations. The existing numerical techniques used in the literature, such as the Adomian decomposition method, the Galerkin method, and others, suffer from the restrictive assumptions that are used to handle nonlinear terms. The VIM has no specific requirements, such as linearization, small parameters, and Adomian polynomials for nonlinear operators. Another important advantage is that the VIM method is capable of greatly reducing the size of calculation while still maintaining high accuracy of the numerical solution. This powerful technique handles both linear and nonlinear equations in a unified manner without any need for the so-called Adomian polynomials that we need if the Adomian decomposition method is applied. Having determined the Lagrangian multiplier for any equation, the approximations are readily obtained.

3 Convergence analysis

In this section, we will show the convergence of the variational iteration procedure given in Eq. (2.8). The scheme (2.10) is simpler and follows in a similar fashion.

Theorem 1 *The sequences* $\{u_n(x)\}_{n=1}^{\infty}$, *where* $x \in [0, 1]$, *defined by* (2.8) *with* $u_0(x) = a + bx$ *(a and b are real constants) converges to the exact solution,* $y(x)$, *of problem* (1.3)–(1.2).

Proof By subtracting $u(x)$ from both sides of (2.8), the scheme can be rewritten as

$$u_{n+1}(x) - u(x) = u_n(x) - u(x) + \int_0^x \lambda(s)\left[(u_n - u)_{ss} + u_{ss} - (1/\alpha(p_n))\, u_n^{2m+1}(s)\right] ds. \quad (3.1)$$

Here $\lambda = s - x$ and note that the integrand was expressed in terms of $u_n - u$. Since $u(x)$ is the exact solution of (1.3), therefore the term u_{ss} in the integrand can be replaced by $(1/\alpha(p_n))\, u_n^{2n+1}(s)$. Upon letting $E_n(x) = u_n(x) - u(x)$, Eq. (3.1) becomes

$$E_{n+1}(x) = E_n(x) + \int_0^x \lambda(s) \times \left[(E_n(s))_{ss} + (1/\alpha(p_n))\, u^{2m+1}(s) - (1/\alpha(p_n))\, u_n^{2m+1}(s)\right] ds. \quad (3.2)$$

Integrating the first term in the integrand twice by parts we have

$$E_{n+1}(x) = E_n(x) + \lambda(x)\,(E_n)_s\,(x) - \lambda'(x)E_n(x) + \int_0^x \lambda''(s)E_n(s)\, ds + \int_0^x \lambda(s)(1/\alpha(p_n))\left(u^{2m+1}(s) - u_n^{2m+1}(s)\right) ds. \quad (3.3)$$

Upon using the three stationary conditions (2.6) into the latter equation we obtain

$$E_{n+1}(x) = \int_0^x \lambda(s)(1/\alpha(p_n))\left(u^{2m+1}(s) - u_n^{2m+1}(s)\right) ds. \quad (3.4)$$

Operating with the L^2-norm on both sides of the last equation we get

$$\begin{aligned}\|E_{n+1}(x)\|_{L^2} &\le |1/\alpha(p_n)| \int_0^x \|\lambda(s)\|_{L^2} \|u_n^{2m+1}(s) - u^{2m+1}(s)\|_{L^2}\, ds \\ &\le \|\lambda(s)\|_\infty |1/\alpha(p_n)| \int_0^x \|u_n^{2m+1}(s) - u^{2m+1}(s)\|_{L^2}\, ds, \quad (3.5)\end{aligned}$$

where $\|\lambda(s)\|_\infty = \max_{s\in[0,1]} |\lambda(s)|$. Clearly $\lambda(s)$ is bounded since

$$\|\lambda(s)\|_\infty = \|x - s\|_\infty \le \|x\|_\infty + \|s\|_\infty \le 1 + 1 = 2.$$

Applying the Mean Value Theorem to the integrand in (3.6) we have

$$\begin{aligned}\|E_{n+1}(x)\|_{L^2} &\le \|\lambda(s)\|_\infty |1/\alpha(p_n)| \int_0^x (2m+1)\ \|\overline{u}^{2m}(s)\|_{L^2}\ \|u_n(s) - u(s)\|_{L^2}\, ds \\ &\le (2m+1)\ \|\lambda(s)\|_\infty |1/\alpha(p_n)| \int_0^x \|\overline{u}^{2m}(s)\|_{L^2}\ \|E_n(x)\|_{L^2}\, ds. \end{aligned} \quad (3.6)$$

Let

$$L = \max_{s\in[0,1]} |\lambda(s)|, \quad S_n = \max_{s\in[0,1]} |1/\alpha(p_n)|, \quad \text{and} \quad P = \max_{s\in[0,1]} |\overline{u}(s)|.$$

Then, from inequality (3.6) we get

$$\|E_{n+1}(x)\|_{L^2} \le (2m+1) L S_n P^{2m} \int_0^x \|E_n(s)\|_{L^2}\, ds. \quad (3.7)$$

Proceeding by induction and letting $L_n = (2m+1) L S_n P^{2m}$, we get

$$\begin{aligned}\|E_1(x)\|_{L^2} &\le L_0 \int_0^x \|E_0(s)\|_{L^2}\, ds \le L_0\ \|E_0(s)\|_\infty \int_0^x ds = L_0\ \|E_0(s)\|_\infty\, x, \\ \|E_2(x)\|_{L^2} &\le L_1 \int_0^x \|E_1(s)\|_{L^2}\, ds \le L_0 L_1\ \|E_0(s)\|_\infty \int_0^x s\, ds = L_0 L_1\ \|E_0(s)\|_\infty\, \frac{x^2}{2}, \\ &\dots \\ \|E_{n+1}(x)\|_{L^2} &\le L_n \int_0^x \|E_n(s)\|_{L^2}\, ds \le \left(\Pi_{i=0}^n L_i\right)\ \|E_0(s)\|_\infty \int_0^x \frac{s^n}{n!}\, ds \\ &= \left(\Pi_{i=0}^n L_i\right)\ \|E_0(s)\|_\infty\, \frac{x^{n+1}}{(n+1)!}. \end{aligned} \quad (3.8)$$

Here $\|E_0(s)\|_\infty = \max_{s\in[0,1]} |E_0(s)|$ and $\Pi_{i=0}^{n} L_i = L_0 L_1 ... L_n$.

Since $\|E_0(s)\|_\infty = \|u_0(x) - u(x)\|_\infty = \|a + bx - u(x)\|_\infty$ and $x \in [0, 1]$, thus

$$\begin{aligned} \|E_0(s)\|_\infty &\le \|a + bx\|_\infty + \|u(x)\|_\infty = |a| + |b| + \|u(x)\|_\infty \\ &= |a| + |b| + \max_{x\in[0,1]} |u(x)|. \end{aligned} \tag{3.9}$$

Note that $u(x)$ belongs to $C^2[0, 1]$ since it is the exact solution of Eq. (1.1) hence it is bounded. Letting $c = \max_{x\in[0,1]} |y(x)|$ we have from (3.9) and (3.8):

$$\begin{aligned} \|E_{n+1}(x)\|_{L^2} &\le \left(\Pi_{i=0}^{n} L_i\right) (a + b + c) \frac{x^{n+1}}{(n+1)!} \\ &\le L_*^n \ (a + b + c) \frac{x^{n+1}}{(n+1)!} = d \ \frac{(L_* \ x)^{n+1}}{(n+1)!}, \end{aligned} \tag{3.10}$$

where $L_* = \max_{0\le i\le n} L_i$ and $d = (a + b + c)/L_*$. Note that for $x \in [0, 1]$ the sequence $\left\{ d \ \frac{(L_* \ x)^{n+1}}{(n+1)!} \right\}$ converges uniformly to 0 as n tends to infinity. Hence by (3.10) it follows that $\|E_{n+1}(x)\|_{L^2}$ converges to 0 which means $u_n(x)$ converges uniformly to $u(x)$. □

4 Numerical test examples

A modified version of the VIM is implemented to obtain numerical solutions to the class of nonlinear, nonlocal elliptic boundary value problems. Five examples are discussed and the results are contrasted with existing exact solutions. The numerical experiments show that the procedure is accurate and converges fast.

Example 1 Consider the following special case of Eq. (1.1) with $\alpha(q) = q^{1/3}$ and $f = -\frac{6}{\sqrt[3]{4}} x$:

$$\left(\int_0^1 u(t) \ dt \right)^{1/3} u''(x) = \frac{6}{\sqrt[3]{4}} \ x, \tag{4.1}$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = 1. \tag{4.2}$$

Problem (4.1)–(4.2) has the exact solution $u(x) = x^3$.

Table 1 VIM applied to Example 1 using the approximation u_{10}

x	Exact solution	VIM solution	Absolute error
0.1	.001	.0010252914	2.5(−5)
0.2	.008	.0080490501	4.9(−5)
0.3	.027	.0270697430	7.0(−5)
0.4	.064	.0640858376	8.6(−5)
0.5	.125	.1250958009	9.6(−5)
0.6	.216	.2160981001	9.8(−5)
0.7	.343	.3430912024	9.1(−5)
0.8	.512	.5120735751	7.4(−5)
0.9	.729	.7290436852	4.4(−5)
1.0	1.00	1.000000000	0.0

The correction functional for (4.1) is of the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left[(u_n(s))_{ss} - \frac{6}{\sqrt[3]{4}}\left(\int_0^1 u_n(x)\,dx\right)^{-1/3} s\right] ds. \tag{4.3}$$

According to the boundary conditions, it is sensible to take the initial iterate as

$$u_0(x) = hx, \tag{4.4}$$

where h is a parameter to be found via matching the boundary condition at $x = 1$ with the variational approximation u_n. The higher successive approximations are lengthy expressions so we only list the first two.

$$\begin{aligned} u_1(x) &= hx + \frac{1}{\sqrt[3]{2h}}\,x^3, \\ u_2(x) &= hx + \frac{\sqrt[3]{2}}{\sqrt[3]{4h + \sqrt[3]{4/h}}}\,x^3. \end{aligned} \tag{4.5}$$

The first six approximations were computed. A comparison between the numerical solution obtained by VIM using $u_{10}(x)$ and the exact solution is depicted in Table 1. The value of h for such a case was found to be $h = 0.0002554690$.

Example 2 Consider the following special case of Eq. (1.1) with $\alpha(q) = q^2$ and $f = -\frac{3}{4}\cos\left(\frac{2\pi}{3}x\right)$:

$$-\left(\int_0^1 u(t)\,dt\right)^2 u''(x) = -\frac{3}{4}\cos\left(\frac{2\pi}{3}x\right), \tag{4.6}$$

subject to the boundary conditions

$$u(0) = 1, \quad u(1) = -1/2. \tag{4.7}$$

Problem (4.6)–(4.7) has the exact solution $u(x) = \cos\left(\frac{2\pi}{3}x\right)$.

The correction functional for (4.6) is of the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left[(u_n(s))_{ss} + \frac{3}{4}\left(\int_0^1 u_n(x)\,dx\right)^{-2}\cos\left(\frac{2\pi}{3}s\right)\right]ds. \tag{4.8}$$

According to the boundary conditions, we may choose the initial iterate as $u_0 = 1+hx$, however it was found that the convergence is very slow. Instead we altered the first approximation and used:

$$u_0(x) = 1 + hx^2, \tag{4.9}$$

where h is a parameter to be determined as in Example 1. The convergence was extremely fast and to achieve high accuracy it suffices to list the first one.

$$u_1(x) = \frac{1}{16\pi^2\left(h^2+6h+9\right)} \times \left[16\pi^2h^2 + 96\pi^2h + 144\pi^2 - 243 + 243\cos\left(\frac{2\pi}{3}x\right)\right]. \tag{4.10}$$

The value of h for such a case was found to be $h = -1.34601331373463$. Alternatively, in order to speed up the convergence, we selected the following initial iterate:

$$u_0(x) = \cos(hx), \tag{4.11}$$

which satisfies the condition at $x = 0$ while h is a parameter to be determined using the second boundary condition at $x = 1$. Again, only one step of the VIM was needed to obtain a highly accurate solution, which is

$$u_1(x) = \frac{1}{\pi^2\sin^2 h}\left[3h^2\cos(2\pi x/3) - \pi^2\cos^2 h + \pi^2 - 3h^2\right]. \tag{4.12}$$

The value of h for such a case was found to be $h = -2.09439510239319$. A comparison between the numerical solution obtained by VIM, using the above two choices of the initial estimate $u_0(x)$, and the exact solution is reported in Table 2. Note that both choices of u_0 matches with the exact solution using only one step of the VIM.

Example 3 Consider the following special case of Eq. (1.1) with $\alpha(q) = (1+q)^2$ and $f = \frac{49}{18}$:

$$-\left(1 + \int_0^1 u(t)\,dt\right)^2 u''(x) = \frac{49}{18}, \tag{4.13}$$

subject to the boundary conditions

Table 2 VIM applied to Example 2 using two different initial approximations u_0

x	Exact solution	VIM solution u_1 ($u_0 = 1 + hx^2$)	VIM solution u_1 ($u_0 = \cos(hx)$)
0.1	.978147600733806	.978147600733793	.978147600733810
0.2	.913545457642601	.913545457642589	.913545457642606
0.3	.809016994374948	.809016994374937	.809016994374953
0.4	.669130606358860	.669130606358850	.669130606358864
0.5	.499999999999998	.499999999999995	.500000000000006
0.6	.309016994374954	.309016994374945	.309016994374955
0.7	.104528463267660	.104528463267655	.104528463267661
0.8	−.104528463267647	−.104528463267649	−.104528463267644
0.9	−.309016994374942	−.309016994374938	−.309016994374937
1.0	−.499999999999995	−.499999999999986	−.499999999999988

$$u(0) = 0, \quad u(1) = 0. \tag{4.14}$$

Problem (4.13)–(4.14) has the exact solution $u(x) = x(1 - x)$.

The correction functional for this case is of the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x (s - x)\left[\left(1 + \int_0^1 u_n(x)\,dx\right)^2 (u_n(s))_{ss} + \frac{49}{18}\right] ds. \tag{4.15}$$

According to the boundary conditions, we may choose the initial iterate to be

$$u_0(x) = hx, \tag{4.16}$$

where h is a parameter to be found via manipulating the boundary condition at $x = 1$. The higher successive approximations are lengthy expressions so we only list the first two.

$$\begin{aligned} u_1(x) &= hx - \frac{49}{36}x^2, \\ u_2(x) &= hx - \frac{972503}{419904}x^2 + \frac{2891}{3888}x^2h + \frac{49}{144}x^2h^2. \end{aligned} \tag{4.17}$$

The first six approximations were computed. The value of h for such a case was found to be $h = 1.003912755$. A comparison between the numerical solution obtained by VIM using $u_6(x)$ and the exact solution is depicted in Table 3.

Example 4 Consider the following special case of Eq. (1.3) with $\alpha(q) = 1/q$:

$$-\frac{1}{q}u''(x) + \frac{3}{4(2\sqrt{2} - 2)}\, u^5 = 0, \tag{4.18}$$

Table 3 VIM applied to Example 3 using the approximation u_6

x	Exact solution	VIM solution	Absolute error
0.1	.09	.0903492755	3.5(−4)
0.2	.16	.1606145510	6.1(−4)
0.3	.21	.2107958265	8.0(−4)
0.4	.24	.2408931020	3.8(−4)
0.5	.25	.2509063775	9.1(−4)
0.6	.24	.2408356530	8.4(−4)
0.7	.21	.2106809285	6.8(−4)
0.8	.16	.1604422040	4.4(−4)
0.9	.09	.0901194795	1.2(−4)
1.0	.00	−.000287245	2.9(−4)

where

$$q = \int_0^1 u(t)\, dt,$$

and subject to the boundary conditions

$$u(0) = 1, \quad u(1) = \sqrt{2}/2. \tag{4.19}$$

Problem (4.18)–(4.19) has the exact solution $u(x) = 1/\sqrt{1+x}$.

The correction functional for this case becomes:

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left[(u_n(s))_{ss} + \frac{3q}{4(2\sqrt{2}-2)}\, u_n^5(s)\right] ds, \tag{4.20}$$

where q is given above. According to the boundary conditions, we may choose the initial iterate to be

$$u_0(x) = 1 + hx, \tag{4.21}$$

where h is a parameter to be found via manipulating the second boundary condition at $x = 1$. The higher successive approximations are lengthy expressions so we only list the first one.

$$\begin{aligned} u_1(x) = 1 + \frac{1}{224(\sqrt{2}-1)} \Big[& h^6x^7 + 2h^5x^7 + 7h^5x^6 + 14h^4x^6 + 21h^4x^5 \\ & + 42h^3x^5 + 35h^3x^4 + 70h^2x^4 + 35h^2x^3 + 70hx^3 \\ & + 224\sqrt{2}hx + 21hx^2 - 224ht + 42t^2 \Big]. \end{aligned} \tag{4.22}$$

The first three approximations were computed. The value of h was found to be $h = -0.4990199339$. A comparison between the numerical solution obtained by VIM using $u_3(x)$ and the exact solution is depicted in Table 4.

Table 4 VIM applied to Example 4 using the approximation u_3

x	Exact solution	(VIM) solution	Absolute error
0.1	.9534625894	.9535746906	1.1(−4)
0.2	.9128709292	.9130017937	1.3(−4)
0.3	.8770580194	.8773763380	3.2(−4)
0.4	.8451542545	.8454589061	3.0(−4)
0.5	.8164965812	.8167992837	3.0(−4)
0.6	.7905694151	.7909250558	3.6(−4)
0.7	.7669649889	.7672832340	3.2(−4)
0.8	.7453559928	.7457397214	3.8(−4)
0.9	.7254762502	.7257126136	2.3(−4)
1.0	.7071067814	.7072857547	1.8(−4)

Table 5 VIM applied to Example 5 using the approximation u_2

x	Exact solution	(VIM) solution	Absolute error
0.1	.9534625894	.9533501404	1.1(−4)
0.2	.9128709292	.9126557645	2.2(−4)
0.3	.8770580194	.8767665261	2.9(−4)
0.4	.8451542545	.8448277743	3.3(−4)
0.5	.8164965812	.8161831014	3.1(−4)
0.6	.7905694151	.7903129070	2.6(−4)
0.7	.7669649889	.7667949063	1.7(−4)
0.8	.7453559928	.7452783290	7.8(−5)
0.9	.7254762502	.7254667949	9.5(−6)
1.0	.7071067814	.7071067678	1.2(−8)

Example 5 Consider the following special case of Eq. (1.3) with $\alpha(q) = q$:

$$-qu''(x) + \frac{3(2\sqrt{2}-2)}{4}\, u^5 = 0, \tag{4.23}$$

where

$$q = \int_0^1 u(t)\, dt,$$

and subject to the boundary conditions

$$u(0) = 1, \quad u(1) = \sqrt{2}/2. \tag{4.24}$$

Problem (4.23)–(4.24) has the exact solution $u(x) = 1/\sqrt{1+x}$.

The correction functional for this case becomes:

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left[(u_n(s))_{ss} - \frac{3(2\sqrt{2}-2)}{4q}\, u_n^5(s)\right] ds, \tag{4.25}$$

where q is given above. According to the boundary conditions, we may choose the initial iterate to be

$$u_0(x) = 1 + hx, \tag{4.26}$$

where h is a parameter to be found via manipulating the second boundary condition at $x = 1$. The higher successive approximations are lengthy expressions so we only list the first one.

$$\begin{aligned} u_1(x) = \frac{1}{14(h+2)} \Big[&\sqrt{2}\, h^5x^7 - h^5x^7 + 7\sqrt{2}\, h^4x^6 - 7h^4x^6 + 21\sqrt{2}\, h^3x^5 \\ &- 21h^3x^5 + 35\sqrt{2}\, h^2x^4 - 35h^2x^4 + 35\sqrt{2}\, hx^3 - 35hx^3 \\ &+ 21\sqrt{2}\, x^2 + 14h^2x + 28hx - 21x^2 + 14h + 28 \Big]. \end{aligned} \tag{4.27}$$

The first two approximations were computed. The value of h was found to be $h = -0.5011368413$. A comparison between the numerical solution obtained by VIM using $u_3(x)$ and the exact solution is depicted in Table 5.

5 Conclusion

A modified version of the VIM was presented and successfully employed for obtaining numerical solutions to a class of nonlinear, nonlocal, elliptic boundary value problems. The convergence of the scheme was formally verified. The algorithm is efficient, practical, and effective when compared with existing techniques. Numerical experiments for the test examples are plausible and the results of the examples that were examined confirm the accuracy, efficiency and fast convergence of the strategy. Furthermore, the approach does not require imposing limiting assumptions that could change the physical structure of the solution.

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